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The music of Charles Fefferman

by

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Abstract. We analyze the mathematical work of Charles Fefferman, on the occasion of his prize *Premio Fronteras del Conocimiento en Ciencias Básicas*, in its 2022 edition. We present a comparative introspective of the creative process through a musical divertimento.

I first met Charles Fefferman (Charlie) in the fall of 1985, upon my arrival at Princeton University, where I went to do my PhD. As a first year student, the tradition at Princeton indicates that our job was not to do research or even learn new mathematics; John Mather, the graduate director that year, made it very clear in his introductory speech: the main task we were given was to go, at exactly 3:30 pm, every single day, to the department common room, for tea. If someone thinks this was a frivolous statement, or simply an exaggeration, I must say that it was actually a metaphor; the message was that we ought to spent our first year getting to know each other in the Department, aside from, of course, taking courses and getting started with our research. A main aspect of that advice was to take courses with the objective of knowing what each Professor was like, with their idiosyncrasies and individual sense of genius.

In the case of Charlie, I already had some prior referential knowledge of his work—young prodigy, adult genius—and knew of one of his most notable results, his proof ([7]), independent of the earlier one by Carleson ([1]), of the pointwise convergence of Fourier series (we refer the interested reader to the appendix for relevant notation, definitions and background information):

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{2\pi n x} = f(x), \quad \text{almost everywhere in } x.$$

This was a question that went back to the second half of the 19th century, as a consequence of the work of Fourier ([12]) and which highlighted important shortcomings of the mathematics of the time; it gave rise to the creation of the Lebesgue measure and the establishment of the δ-ε arguments by Cauchy, who formalized the definition of the limit. It is also responsible for the work of Cantor and the formal logic of Frege, Whitehead and Russell; in short, the problem is at the root of mathematics as we know them today.

But the original problem, in extension given to it by the mathematicians of the 20th century, remained open until it was solved by Carleson in 1966 ([1]), the year that Charlie entered the graduate program at Princeton University, as a seventeen year old. Charlie built his own proof seven years later, as a 24 year old, the same age that Mozart had when he composed his *34th Symphony* and his opera *Idomeneo*, the young Mozart of Salzburg before he would move to Vienna; for this reason, I arrived in Princeton wondering if there was any analogy between the music of Mozart and the mathematics of Charlie, with a personal challenge to find traces of the *Jupiter Symphony* or *The Magic Flute* in his way to understand mathematics.

It all became clear one day when I was sitting in his course on the stability of matter, a topic he had just started to revolutionize as I arrived in Princeton with his paper ([8]) on the mathematics of atoms and molecules, which by the way was dedicated to his favourite violin player: his wife Julie. There, he presented the equation

$$\frac{1}{|x|} = \frac{1}{\pi} \iint_{R>0, z\in\mathbb{R}^3} \left\{ \begin{matrix} 1 & \text{if } x, 0 \in B(z,R) \\ 0 & \text{otherwise} \end{matrix} \right\} \frac{dz\, dR}{R^5}, \qquad (1)$$

which he joyfully proved with the observation that both sides behave in the same way under dilations and rotations, which forces them to be constant multiples of each other, and the constant $\frac{1}{\pi}$ is something you just calculate. That formula allowed him to express the potential energy of the atomic hamiltonian as a childish count of the number of particles in balls. At that time I had no doubt: Charlie was Beethoven, and his equation (1) corresponded with the first bar of his *Fifth Symphony*

which was followed was the second bar

for the relativistic kinetic energy

$$\left\langle (-\Delta)^{1/2} u, u \right\rangle = c \iint_{R>0, z\in\mathbb{R}^3} \int_{x,y\in B(z,R)} |u(x) - u(y)| \frac{dz\, dR}{R^8},$$

with a similar short proof as before. The result was a fantastic combination of these two themes, with its developments, expositions, variations and recapitulations, which ended in the splendid symphony of the relativistic stability of matter, proved jointly with Rafael de la Llave ([11]):

$$\langle H_{Z,M,N}\psi, \psi \rangle \geq -C_Z \cdot (M+N).$$

We refer the interested reader to the appendix for relevant definitions of the operators $H_{Z,M,N}$ and constants C_Z in this expression, as well as other useful details.

I continued with my PhD, happy and convinced I was under the supervision of Beethoven, until a few years later, searching for inspiration to write my thesis, I had the occurrence to read Charlie's, conveniently located in the basement of Fine Hall, where the Princeton mathematics department was. There I found a manuscript written on a typewriter with hand-made drawings in which Charlie has established the foundations for a new approach to harmonic analysis, with which he would establish, just a few months later ([5]), a famous conjecture of that time, related to the multiplier problem for the ball, an operator defined by truncating the Fourier transform of a function by the characteristic function of the unit ball,

$$\widehat{Tf}(\xi) = \chi_{B(0,1)}(\xi) \cdot \hat{f}(\xi).$$

If one replaces the unit ball by a rectangle, the operator is essentially the Hilbert transform, the cornerstone of the School of Chicago, which was well known to be bounded in all $L^p(\mathbb{R})$ spaces, for $1 < p < \infty$. The thinking was that the ball is not so different from a rectangle, and therefore the corresponding operator should also be bounded in L^p, $1 < p < \infty$. The case $p = 2$ was obvious, but the problem remained open for $p \neq 2$. Charlie proved, in his famous article [5], published in 1970, that this was not the case: the multiplier for the ball was not bounded in *any* L^p, except in the trivial case $p = 2$. This result was revolutionary and totally unexpected, and he proved it just a few months after his thesis, which I was about to start to read.

My surprise was enormous as I read his introduction, a superb document, written by a teenager, that explained in a simple but deep manner which were the complications of the problem, how to approach them and why previous attempts had not worked. Reading these few pages in the introduction, I could not stop visualizing the members of the Chicago School, one of the strongest sources of mathematical thinking of the twentieth century, as simple Salieris. Of course this is not true, in fact, much of the wisdom and style which emerged from those pages had the DNA of Elias Stein, Charlie's thesis director, who was portrayed by Charlie himself in his famous tribute ([10]). However, at that precise moment, as I was reading those pages, I had no doubt: Charlie was Mozart.

One could also argue that Charlie was Johann Sebastian Bach. In fact, his most celebrated result ([6]), which earned him the Fields medal in 1978, and is expressed with the simple and now famous equation

$$H^1 = \mathrm{BMO}^*. \tag{2}$$

In line with the explanations that Charlie himself uses when he presents the foundations of this result to the general public, the origin of these mathematical objects was in the two dimensional reality which was the focus of 19th century science, ranging from the understanding of electric fields to the manner in which we can make maps of surfaces such as the earth. From our musical perspective, I want to underline the palindromic elements that (2) shares with Bach's *Musical Offering*:

If you look at the music score from the end, you can see an inverted C-clef, which indicates the informed reader that the writing is actually intended for two voices, one which runs the staff forward and the other one which runs it backwards. In his result [6] of 1971 (see also the brilliant exposition by Stein himself in [3]), Charlie connected two worlds. On the one hand, the world of partial differential equations of John, Nirenberg, Lax, Moser, Nash, DiGiorgi and many others, for whom the spaces BMO of (*Bounded Mean Oscillation*, see the appendix) was a cornerstone in the rigorous treatment of many problems, notably non-linear elasticity. On the other hand, the analysis of the Chicago school, mentioned above, had found the Hardy spaces H^1 as a convenient substitute of L^1 in the study of singular integrals, such as the Hilbert transform, also mentioned earlier. The result in (2) therefore allows a round-trip relationship between partial differential equations and Fourier analysis, reminiscent of the epiphany that Fourier's publication in 1822 ([12]) signified for mathematics, creating those echos so familiar in the organ music of the great composer from Leipzig, and which Elias Stein so well described in his ode to Charlie, published in the *Notices of the AMS* ([4]).

I never imagined, back in 1985, that a decade of intense collaboration with Charlie awaited me; a few years later, having reached my adult life, I had a more solid perspective on his type of mathematics. Over a period of a few years, we wrote together publications which amounted to more than one thousand pages, most of them written by Charlie himself, at lightning speed. For that reason, someone who knew no math or music could compare him to Rossini, the fastest composer of all time, who wrote the 150 000 notes of *The Barber of Seville* in three weeks, and of whom Richard Wagner would say that he had the ability to write narcotic melodies with astonishing ease. Wagner would never say anything nice about anyone, for that reason Charlie, who is a beautiful person, cannot be compared with him; or with Rossini, as Charlie's math is always deep. At a more personal level, he would also not be Leonardo da Vinci who, with his excellent cookbooks would settle a large cooking skill distance between the two.

More recently ([9]), Charlie has been focusing much of his work on the Whitney extension theorems, whose original formulation goes back to 1934 and deals with the extension of functions f defined on an arbitrary set $E \subset \mathbb{R}^n$ to a smooth function in the entire euclidean space, $F \in C^M(\mathbb{R}^n)$. There are many questions to address here: does such an extension exist?; if it does, how small can its C^M norm be?; what is the relationship between f and F? These questions have an academic backdrop of algebraic geometry and complex variables, but viewed in an interpolation/extrapolation context, we suddenly realize their relevance in the postmodern world of data science, similar to the way Stravinsky approached his composition *Pulcinella*, where he had no doubt to develop classical music themes, which by the way created negative reactions from some of his colleagues at the time.

Poincare said that there are two types of mathematicians, the logical ones and the intuitive ones which, from our musical perspective, would translate into those who master the lyrics and those who master the music; Charlie would be both, and therefore would be compared to Schubert and each of his papers would be a superb *lieder*. But perhaps Charlie would be more revolutionary than that; I would argue that, beyond music and lyrics, Charlie would be like Monteverdi, who created a new musical style, bringing together music, literature, stage performance and ballet to reach a new height: the Opera. His first such creation, *L'Orfeo*, unavoidably deals with how music allows the hero charm Charron (Caronte) and rescue his lover Euridice from the underworld, placing the music into centerstage and, like in the case of Charlie, turning his work into something that redeems us.

To end this digression, and aside from possible disagreements with my remarks, either of mathematical or musical nature, I hope that we all agree that the mathematics of Charlie Fefferman deserve a celebration, and the prize *Fronteras del Conocimiento* is a fair tribute. Charlie's footprint is significant and deep and has been already described in numerous publications ([2], [4], and many others). Therefore, I encourage everyone, emulating what Charlie's fist PhD student, Antonio Cordoba once did, to a toast:

¡A su salud, maestro!

Appendix

1. The Fourier coefficients of a function $f : [0,1] \to \mathbb{R}$ are defined as:

$$\hat{f}(n) = \int_0^1 f(x)\, e^{-2\pi i n\cdot x}\, dx, \qquad n \in \mathbb{N},$$

and the Fourier transform of f as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi\cdot x}\, dx, \qquad \xi \in \mathbb{R}^n.$$

2. Stability of matter is expressed mathematically as the inequality

$$\langle H_{Z,M,N}\psi, \psi\rangle \geq -C_Z \cdot (M+N),$$

where

$$H_{Z,M,N} = \sum_{k=1}^{N} (-\Delta_{x_k}) + V_{Z,M,N} \tag{3}$$

is the Hamiltonian of M nuclei of charge Z_i located at y_i with N electrons located at x_j; C_Z is a constant which only depends on the charges Z_i. In (3) the laplacian term represents the kinetic energy; the potential energy is given by the multiplicative operator

$$V_{Z,M,N} = \sum_{j<k} \frac{1}{|x_j - x_k|} + \sum_{j<k} \frac{Z_j\, Z_k}{|y_j - y_k|} - \sum_{j,k} \frac{Z_k}{|x_j - y_k|}.$$

3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is of *Bounded Mean Oscillation* (BMO) when its variation around the mean $f_Q = \fint_Q f(x)\,dx$ is bounded for all cubes $Q \subset \mathbb{R}^n$,

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx \le A < \infty.$$

On the other hand, $H^1(\mathbb{R}^n)$ consists of functions f in $L^1(\mathbb{R}^n)$ such that their Riesz transforms $R_k f$ are also in $L^1(\mathbb{R}^n)$:

$$\widehat{R_k f}(\xi) = -i\,\frac{\xi_k}{|\xi|} \cdot \hat{f}(\xi), \qquad 1 \le k \le n, \quad \xi \in \mathbb{R}^n.$$

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