Olga Ladyzhenskaya and Olga Oleinik: two great women mathematicians of the 20th Century

Susan Friedlander and Barbara Keyfitz

This short article celebrates the contributions of women to partial differential equations and their applications. Although many women have made important contributions to this field, we have seen the recent deaths of two of the brightest stars –Olga Ladyzhenskaya and Olga Oleinik– and in their memory we focus on their work and their lives.

The two Olgas had much in common and were also very different. Both were born in the 1920s in the Soviet Union and grew up during very difficult years and survived the awful death and destruction of the 2nd world war. Shortly after the war they were students together at Moscow State University where they were both advised by I.G. Petrovsky, whose influence on Moscow mathematics at the time was unsurpassed. Both were much influenced by the famous seminar of I.M.Gelfand and both young women received challenging problems in PDE from Gelfand. In 1947 both Olga's graduated from Moscow State University and then their paths diverged. Olga Oleinik remained in Moscow and continued to be supervised by Petrovsky. Her whole career was based in Moscow and after receiving her PhD in 1954 she became first a professor and ultimately the Head of the department of Differential Equations at Moscow State University. Olga Ladyzhenskaya moved in 1947 to Leningrad and her career developed at the Steklov Institute there. Like Oleinik, her mathematical achievements were very influential and as a result of her work Ladyzhenskaya overcame discrimination to become the uncontested leader of the Leningrad school of PDE.

It is our understanding that the personalities of the two Olga's were rather different, although they were both women of great strength and determination. Oleinik was a member of the academic establishment with all that this implied in the Soviet system: Ladyzhenskaya, whose father was arrested and killed as a "class enemy" was outside the establishment and at times openly critical of the system. However, in both cases their superb mathematics well merited the ultimate "seal of approval" of the establishment, namely election to membership in the Russian Academy of Sciences.

1. Olga Ladyzhenskaya

Olga Alexandrovna Ladyzhenskaya was born on March 7th, 1922 in the rural Russian town of Kologriv and died in her sleep on January 12th, 2004 in St. Petersburg, Russia at the age of 81. She left a wonderful legacy for mathematics in terms of her fundamental results connected with partial differential equations and her "school" of students, collaborators and colleagues in Russia. In a life dedicated to mathematics she overcame personal tragedy arising from the cataclysmic events of 20th century Russia to become one of that country's leading mathematicians.

Olga Alexandrovna Ladyzhenskaya

In 1939 she passed the entrance exams for Leningrad University, which at the time was the best university in the Soviet Union. However she was denied a place as an undergraduate at the university because, although she was an exceptionally gifted young girl, she was one whose father disappeared in Stalin's gulag. Her father had taught mathematics at a high school and it was her father who introduced Olga at an early age to mathematics and calculus. In 1937 her father was arrested and later killed by the NKVD, the foreruner of the KGB. Life then became extremely difficult for his family who lived in disgrace and poverty as the family of a "class enemy ". With help from friends Olga finally became a student at Moscow State University in 1943 and she graduated in 1947. There I.G.Petrovsky was her advisor and she was also strongly influenced by I.M.Gelfand.

Olga married a Leningrad mathematician, A.A. Kiselev in 1947 and became a graduate student at Leningrad State University. Her advisors were

S.L.Sobolev and V.I.Smirnov. Her Ph.D thesis, defended in 1949, was a breakthrough in the theory of PDE and later developments concerning weak solutions to initial boundary value problems became important concepts in mathematical physics. From 1947 onwards she was very actively involved in the Leningrad seminar on mathematical physics which brought together many mathematicians working in PDE and their applications. She remained one of the leaders of the seminar until her death.

For most of her professional career Olga was a member of the Steklov Institute in Leningrad/ St Petersburg (called LOMI and now called POMI). She rose to become one of the most distinguished and influential members of POMI. She was elected to the Russian National Academy of Sciences (a corresponding member in 1981 and a full member in 2002). Among her prizes was the Kovaleskaya Prize of the Russian Academy.

Her mathematical achievements were honoured in many countries. She was a foreign member of several academies including the Leopoldina, the oldest German academy. Among other offices, she was President of the Mathematical Society of St. Petersburg and as such a successor of Euler. Recently she was awarded the degree of "Doctor Honoris Causa" by the University of Bonn and an excellent description of her achievements can be found in the "laudatio" given for this occasion by M.Struwe [1].

Ladyzhenskaya made deep and important contributions to the whole spectrum of partial differential equations and worked on topics that ranged from uniqueness of solutions of PDE to convergence of Fourier series and finite difference approximation of solutions. She used functional analytic techniques to treat nonlinear problems using Leray-Schauder degree theory and pioneered the theory of attractors for dissipative equations. Developing ideas of De Giorgi and Nash, Ladyzhenskaya and her coauthors gave the complete answer to Hilbert's 19th problem concerning the dependence of the regularity of the solution on the regularity of the data for a large class of second order elliptic and parabolic PDE. She published more than 250 articles and authored or co authored seven monographs and textbooks. Her very influential book "The Mathematical Theory of Viscous Incompressible Flow" which was published in 1961 has become a classic in the field. Her main mathematical "love" was the PDE of fluid dynamics, particularly the Navier-Stokes equation. This equation has a long and glorious history but remains extremely challenging: for example, the issue of existence of physically reasonable solutions to the Navier-Stokes equations in 3 dimensions was chosen as one of the seven millennium "million dollar" prize problems of the Clay Mathematical Institute. (for details, see the problem description by Fefferman [2]). The 3 dimensional problem remains open to this day, although it was in the 1950's that Ladyzhenskaya obtained the key result of global unique solvability of the initial boundary problem for the 2 dimensional Navier-Stokes equation. She continued to obtain influential results and raise stimulating issues in fluid dynamics, even up to the days before her death. Ladyzhenskaya considered fluid dynamics not only in the framework of the Navier-Stokes equations. She explored alternative models for such challenging issues as turbulence and this led her to study the notion of an attractor for infinite dimensional dynamical systems. In this connection she opened a new direction in the theory of PDE, namely "stability in the large".

Further details concerning Ladyzhenskaya's significant mathematical achievements can be found in the forthcoming memorial article in the AMS Notices [3] and in the volumes published in honor of her 80th birthday [4]. Olga was a woman of great charm and beauty. She was part of a circle of Russian intellectuals of world wide fame including A.Solzhenitsyn, A.Akhmatova and J.Brodsky. Her friends, colleagues and collaborators, G.Seregin and N.Uraltseva tell us that it was not only Olga's scientific results, though truly deep and fundamental, but also her personal integrity and energy that played an especial role in her contribution to mathematics.

SELECTED HONORS OF OLGA LADYZHENSKAYA

1969 The State Prize of the USSR
1985 Elected a foreign member of the Deustche Akademie Leopoldina
1989 Elected a member of the Accdemia Nazionale dei Lincei
1990 Elected a full member of the Russian Academy of Sciences
2002 Awarded the Great Gold Lomonosov Medal of the Russian Academy
2002 Doctoris Honoris Causa, University of Bonn

Selected Publications of Olga Ladyzhenskaya

[1] WITH N.URALTSEVA, *L*inear and Quasilinear Elliptic Equations, Nauka, Moscow 1964 ; Engl trans, Academic Press, New York 1968

[2] *T*he Mathematical Theory of Viscous Fluids, Fizmatgiz, 1961; Engl tran , Gordon and Breach, New York 1969

[3] WITH A.KISELEV, *On the existence and uniqueness of the solution to the non-stationary problem for a viscous incompressible fluid.* Izv.Akad.Nauk SSSR Ser Mat 21 , 665-680 , 1957.

[4] *Solution in the large of boundary value problems for the Navier-Stokes equations in 2 space variables.* Comm Pure Appl Math 12, 427-433 1959

[5] WITH V.SOLINNIKOV AND N.URALTSEVA, *L*inear and Quasilinear Equations of Parabolic Type, Nauka, Moscow 1967; Eng trans Transl Math Monographs, 23, AMS, 1968

[6] *Attractors for Semigroups and Evolution Equations*, Lezioni Lincei, Cambridge Univ Press, 1991.

2. Olga Oleinik

Olga Arsenievna Oleinik was born in the Ukraine on July 2, 1925 and died of cancer on October 13, 2001.

Olga Arsenievna Oleinik

She obtained her PhD from Moscow State University, where she spent her career, in 1954, a student of Ivan Petrovsky, one of the founders of the modern theory of partial differential equations (PDE). As Petrovsky's successor, she built a strong team in PDE, and from the start of her career she also explored applications in elasticity, and in several aspects of fluid flow, including compressible gas dynamics and the filtration equation of flow in porous media.

Near the beginning of her career, she contributed greatly to the theory of hyperbolic conservation laws, then in its infancy. Conservation laws are nonlinear partial differential equations of the form

$$u_t + f(u)_x = 0\,. \tag{1}$$

Here u is a scalar or vector quantity, and f a corresponding flux function. The equation (1) expresses conservation of the components of u –typically mass, momentum and energy. The system is hyperbolic when the Jacobian of the flux, df, has a full set of real eigenvalues and eigenvectors. When $f(u) = Au$ for a matrix A, the system is linear and its solutions, including weak solutions, are well understood from linear theory.

The wartime work of Courant, Friedrichs and others had established the necessity of finding a nonlinear theory for weak solutions, as classical hyperbolic theory could not explain the spontaneous formation of shocks, the fact that nonlinear equations gave rise to discontinuities that did not propagate along characteristics, or the ensuing questions about lack of uniqueness. In addition, global existence theorems were lacking, and even the correct function spaces in which to seek solutions were unknown, despite the fact that these equations underlay the technology of explosions and the new field of supersonic flight. The work of Oleinik changed this. She proved existence of weak solutions to the scalar equation (1), for general flux functions, showing they were limits of the perturbed equation

$$u_t + f(u)_x = \epsilon u_{xx} , \tag{2}$$

generalizing work of Hopf. It would be almost 50 years until this result was broadened to systems of conservation laws. In her investigation, Oleinik found the correct space – BV – for solutions. She also developed what is now called the Oleinik entropy condition for uniqueness of solutions of the scalar equation (1). Finally, she proved a uniqueness result for solutions of certain systems, modelled on gas dynamics –this at a time when no existence theorems for systems had yet been proved; the first existence theorem for systems, due to Glimm, appeared shortly after her result. Again, the uniqueness result was not improved for over 30 years.

Oleinik developed fundamental mathematical results in other areas related to classical fluid flow: boundary layer theory (the stability of boundary layers, where viscosity is important only close to the body), and degenerate elliptic equations (motivated by change of type in steady transonic flow). In this last field, termed ‘equations with non negative characteristic form’, she completed and extended work of the Italian school, notably Fichera and Tricomi.

Later in her career, Oleinik turned her attention to diverse other areas: the Stefan problem, in which the mathematical interest is that it provides a free boundary problem for a parabolic equation, and the applications interest is in phase transitions. She also provided the basic theory of weak solutions for the nonlinear degenerate parabolic equation known as the filtration equation. In the 1990s, Oleinik, with Jikov and Kozlov, helped to develop the mathematical theory of homogenization.

In all, her list of publications indexed by Math Reviews includes over 400 items, displaying an astonishing breadth and depth. A memorial article in the Notices remarks on her love of travel, her eagerness to make contacts between Soviet and Western mathematicians, and her loyalty to her friends.

SELECTED HONORS OF OLGA OLEINIK

1981 Honorary Doctorate, University of Rome
1983 Elected an Honorary Member of the Royal Society of Edinburgh
1988 Elected a member of the Academia Nazionale dei Lincei
1990 Elected a full member of the Russian Academy of Sciences

1996 Association for Women in Mathematics Noether Lecturer
She was also awarded the Petrovsky Prize and the Medal of the College de France.

Selected Publications of Olga Oleinik

[1] *Discontinuous solutions of non-linear differential equations.* Uspehi Mat. Nauk (N.S.) 12 1957 no. 3(75), 3–73.

[2] *On the uniqueness of the generalized solution of the Cauchy problem for a non-linear system of equations occurring in mechanics.* Uspehi Mat. Nauk (N.S.) 12 1957 no. 6(78), 169–176.

[3] *Construction of a generalized solution of the Cauchy problem for a quasi-linear equation of first order by the introduction of "vanishing viscosity".* Uspehi Mat. Nauk 14 1959 no. 2 (86), 159–164.

[4] *On Stefan-type free boundary problems for parabolic equations.* 1962/1963 Seminari 1962/63 Anal. Alg. Geom. e Topol., Vol. 1, Ist. Naz. Alta Mat. pp. 388–403 Ediz. Cremonese, Rome.

[5] with E. V. Radkevič, *Second order equations with nonnegative characteristic form. Mathematical analysis*, 1969, pp. 7–252. (errata insert) Akad. Nauk SSSR Vsesojuzn. Inst. Naučn. i Tehn. Informacii, Moscow, 1971.

[6] with V. V Jikov and S. M. Kozlov. *Homogenization of differential operators and integral functionals.* Translated from the Russian by G. A. Yosifian. Springer-Verlag, Berlin, 1994.

References

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[2] C. Fefferman, *Existence and smoothness of the Navier-Stokes equations.* `www.claymath.org/millennium/Navier-Stokes_equations`

[3] S. Friedlander, P. Lax, C. Morawetz, L. Nirenberg, G. Seregin, N. Uraltseva, M. Vishik, "Olga Alexandrovna Ladyzhenskaya, (1922-2004)", *Notices of the AMS*, **51** (2004) 11, 1320–1331.

[4] *Nonlinear Problems in Mathematical Physics and Related Topics: In honor of Prof O.A. Ladyzhenskaya.* Ed. Birman, Hildebrandt, Solonnikov and Uraltseva. Internation. Math. Series , Kluwer/Plenum (Inglés) y Rozhovskaya (Ruso) (2002).

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[6] WILLI JAGER, PETER LAX, CATHLEEN SYNGE MORAWETZ, "Olga Arsenevna Oleinik (1925-2001), *Notices of the AMS*, **50** (2003), 220–223.

[7] J. J. O'CONNOR, E. F. ROBINSON, *"Olga Arsenevna Oleinik"*, `http://www-gap.dcs.st-and.ac.uk/~history/References/Oleinik.html`

Susan Friedlander
Dept of Mathematics, Statistics and Computer science
University of Illinois-Chicago
Chicago Il 60607
USA
email: `susan@math.uic.edu`

Barbara Lee Keyfitz
Director Fields Institute
Research in Mathematical Sciences
222 College Street, 2nd Floor
Toronto, Ontario
M5T 3J1 Canada
email: `bkeyfitz@fields.utoronto.ca`